

# Distortion estimates for barycentric coordinates on Riemannian simplices

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## Abstract

We define barycentric coordinates on a Riemannian manifold using Karcher’s center of mass technique applied to point masses for  $n+1$  sufficiently close points, determining an  $n$ -dimensional Riemannian simplex defined as a “Karcher simplex.” Specifically, a set of weights is mapped to the Riemannian center of mass for the corresponding point measures on the manifold with the given weights. If the points lie sufficiently close and in general position, this map is smooth and injective, giving a coordinate chart. We are then able to compute first and second derivative estimates of the coordinate chart. These estimates allow us to compare the Riemannian metric with the Euclidean metric induced on a simplex with edge lengths determined by the distances between the points. We show that these metrics differ by an error that shrinks quadratically with the maximum edge length. With such estimates, one can deduce convergence results for finite element approximations of problems on Riemannian manifolds.

**Contents:** **1.** Introduction (*Barycentric coordinates, Main results, An application: The Poisson equation, Related approaches using the Karcher Mean, Structure of the paper*). – **2.** Flat Metrics and Barycentric Coordinates (*Parametrizations of Euclidean simplices, Riemannian metrics on Euclidean simplices, Euclidean simplices from geodesic simplices*). – **3.** Barycentric coordinates from center of mass (*Definition of the barycentric coordinate map, Derivatives of the barycentric coordinate map*). – **4.** Estimates (*Preliminary estimates, First derivative estimates, Second derivative estimates*). – **5.** Appendix (*Jacobi Fields with Two Fixed Values, Derivatives of the Squared Distance Function*).

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# 1 Introduction

There are two major approaches to numerical computations on a Riemannian manifold: (1) mapping the smooth manifold to a triangulated manifold and performing computations on Euclidean simplices (see, e.g., [Dzi88, Bar10, HS12]), or (2) performing each computation in a natural chart, such as geodesic (normal) coordinates (e.g., see [Mün07, HT04, CM06]). The advantage of using triangulations is that they provide a global description, independent of coordinates. Normal coordinates, in contrast, allow for interior estimates on metric distortion and curvature inside coordinate charts, but can be difficult to work with globally due to coordinate changes. Here we present a matrimony between these two approaches using Karcher’s center of mass technique.

The goal of the present work is to give a comprehensive treatment of the interior estimates from the perspective of geometric analysis. We will not provide details for applications, most of which follow from our estimates using standard principles from numerical analysis. Many possible applications are given in [Dey13].—For a short overview of related work, see sec. 1.4.

## 1.1 Barycentric coordinates

Riemannian barycentric coordinates are based on the notion of barycentric coordinates in Euclidean space. Let  $\Delta := \text{conv}(e_0, \dots, e_n) \subset \mathbb{R}^{n+1}$  denote the standard simplex, and let  $p_0, \dots, p_n$  be points in Euclidean space  $\mathbb{R}^m$ . The barycentric coordinates of the (possibly degenerate) simplex  $s := \text{conv}(p_0, \dots, p_n)$  are defined by

$$x : \Delta \rightarrow s, \quad x(\lambda) = \sum_{i=0}^n \lambda^i p_i.$$

Notice that since the  $\lambda^i$  sum to 1, the point  $x(\lambda)$  is the minimizer of the function

$$E_\lambda(a) := \sum_{i=0}^n \lambda^i |a - p_i|^2.$$

This construction can be generalized to Riemannian manifolds. To do this we will need a notion of a Riemannian simplex. Let  $p_0, \dots, p_n$  be distinct points in a convex ball  $B$  of a complete Riemannian manifold  $(M, g)$  of dimension  $m$ . Let the pairwise geodesic distances between these given points satisfy  $\mathbf{d}_g(p_i, p_j) \leq h$  for some  $h > 0$ . Let  $g^e$  be the (unique if it exists) flat metric on the standard simplex  $\Delta \subset \mathbb{R}^{n+1}$  such that the induced edge lengths of  $\Delta$  are given by the geodesic distances  $\mathbf{d}_g(p_i, p_j)$ . Below we discuss conditions for when such a flat metric exists. Suppose that for some  $\vartheta > 0$ ,  $g^e$  gives a “ $(\vartheta, h)$ -full” simplex in the sense that the induced volume satisfies  $n! \text{vol}_{g^e}(\Delta) \geq \vartheta h^n$ . Then for fixed  $\vartheta$  and sufficiently small  $h$ , the map (introduced in [GK73])

$$x : \Delta \rightarrow M, \quad \text{defined via} \quad x(\lambda) = \underset{a \in M}{\operatorname{argmin}} \sum_{i=0}^n \lambda^i \mathbf{d}_g^2(p_i, a),$$

is a bijection between  $\Delta$  and a subset  $s$  of  $B$ , called the *Karcher simplex* with respect to vertices  $p_i$ .

Now consider a global triangulation of  $M$  by Karcher simplices. Notice that if  $\lambda \in \Delta$  lies in the facet opposite to the  $i$ 'th vertex of  $\Delta$ , then its component  $\lambda^i$  is zero and  $x(\lambda)$  does not depend on  $p_i$ . Therefore, the flat (Euclidean) simplices can be glued together to give a piecewise flat manifold that is homeomorphic to  $M$ , and  $x$  can be extended to provide a global homeomorphism. The non-degeneracy of these simplices will be assumed in our setting, but is currently investigated in more detail in [DVW15] (see also the previous work in [BDG11]).

We provide estimates on how well the piecewise Euclidean structure,  $g^e$ , of this piecewise flat manifold approximates the smooth Riemannian one,  $g$ , of  $M$ . Notice that on each Karcher simplex the map  $x$  pulls back the Riemannian metric  $g$  on  $M$  to  $\Delta$ . We derive estimates for both first and second derivatives of  $x$  on a simplex. We then derive estimates for the difference  $x^*g - g^e$  as well as for  $\nabla^e x^*g$ , where  $\nabla^e$  denotes the covariant derivative induced by  $g^e$ .

## 1.2 Main results

The tangent space  $T_\lambda\Delta$  of  $\Delta$  can be identified with  $\{v \in \mathbb{R}^{n+1} : \sum v^i = 0\}$ . Let  $p_0, \dots, p_n \in M$  be distinct points inside a convex ball of radius  $h$  in the Riemannian manifold  $(M, g)$  of dimension  $m$ . A set is convex if each pair of points has a unique shortest geodesic that lies entirely in that set.

**Definition 1.** For an arbitrary fixed  $\lambda \in \Delta$  and  $v, w \in T_\lambda\Delta$  define

$$g^e(v, w) = -\frac{1}{2} \sum_{i,j=0}^n \mathbf{d}^2(p_i, p_j) v^i w^j.$$

In Section 2.1 we discuss conditions for when  $g^e$  yields a metric, i.e., when it is positive definite. For now suppose it does. In the following we are interested in how well  $x^*g$  is approximated by  $g^e$ .

In our estimates we require a definition of fullness that quantifies how “thin” a simplex can become with respect to some Riemannian metric.

**Definition 2.** A  $n$ -simplex  $s$  with Riemannian metric  $g$  is  $(\vartheta, h)$ -full if all edges have length less than or equal to  $h$  and

$$n! \operatorname{vol}_g(s) \geq \vartheta h^n,$$

where  $\operatorname{vol}_g(s)$  is the Riemannian volume.

Among Euclidean simplices, the maximal  $\vartheta$  is attained for the equilateral simplex, which shows  $\vartheta \leq \sqrt{n+1}/2^{n/2}$ .

Now let us look at  $x$  from above for the simplest case  $M = \mathbb{R}^{n+1}$ . Here, one has  $dx(v) = \sum v^i X_i|_{x(\lambda)}$  for every  $v \in T_\lambda\Delta$ , where  $X_i$  is the vector field on  $\operatorname{conv}(p_0, \dots, p_n)$  defined by  $X_i = \frac{1}{2} \operatorname{grad} \mathbf{d}^2(\cdot, p_i)$ . This motivates the following definition in the Riemannian setting.

**Definition 3.** For every  $v \in T_\lambda \Delta$  define

$$\sigma(v) = \sum_{i=0}^n v^i X_i|_{x(\lambda)},$$

where  $X_i = \frac{1}{2} \text{grad } \mathbf{d}^2(\cdot, p_i)$  are vector fields on  $x(\Delta) \subset M$ .

The following theorem quantifies how much  $\sigma$  deviates from  $dx$ . Before stating this result, we require some additional notation. We use  $\nabla dx$  to denote the Hessian of  $x$  with respect to the Levi-Civita connection on  $(M, g)$  and the flat connection induced by  $g^e$  on  $\Delta$ . We use  $\|R\|_\infty$  and  $\|\nabla R\|_\infty$  to denote the supremum over the manifold  $M$  of the usual pointwise 2-norm of the Riemannian curvature tensor and its covariant derivative (with respect to the metric  $g$ ), respectively.

Throughout, we let  $C_0 := \|R\|_\infty$ ,  $C_1 := \|\nabla R\|_\infty$ , and we let  $\iota_g$  denote the injectivity radius of  $(M, g)$ . Additionally, when working in a ball of radius  $h$ , we use  $C_{0,1} := C_0 + hC_1$ .

**Theorem 1.** *There exist constants  $\alpha = \alpha(n, \vartheta, C_0, C_1)$ ,  $\beta = \beta(n, C_0)$ , and  $\gamma = \gamma(n, \vartheta, C_0, C_1)$  such that if  $h < \alpha$  and  $(\Delta, g^e)$  is a  $(\vartheta, h)$ -full simplex then*

$$|dx(v) - \sigma(v)|_g \leq \beta h^2 |v|_{g^e}, \quad (1)$$

and

$$|\nabla dx(v, w)|_g \leq \gamma h |v|_{g^e} |w|_{g^e}, \quad (2)$$

for tangent vectors  $v, w \in T_\lambda \Delta$  at any  $\lambda \in \Delta$ .

These estimates can also be interpreted as estimates on the difference between the flat metric  $g^e$  and the pullback of the metric  $g$  by the map  $x$ . We use  $\nabla^e$  to denote the flat connection on  $(\Delta, g^e)$ .

**Theorem 2.** *There exist constants  $\alpha = \alpha(n, \vartheta, C_0, C_1)$ ,  $\beta = \beta(n, \vartheta, C_0)$ , and  $\gamma = \gamma(n, \vartheta, C_0, C_1)$  such that if  $h < \alpha$  and  $(\Delta, g^e)$  is a  $(\vartheta, h)$ -full simplex then*

$$|(x^*g - g^e)(v, w)| \leq \beta h^2 |v|_{g^e} |w|_{g^e}, \quad (3)$$

and

$$|\nabla^e x^*g(u, v, w)| \leq \gamma h |u|_{g^e} |v|_{g^e} |w|_{g^e} \quad (4)$$

for tangent vectors  $u, v, w \in T_\lambda \Delta$  at any  $\lambda \in \Delta$ .

In loose terms, eqn. (3) gives second-order control over  $x$ 's first derivatives, whereas eqn. (4) gives first-order control over the second derivatives.— These theorems follow immediately from Theorems 19, 20, 21, and 22 and are proven in Section 4.

### 1.3 An application: The Poisson equation

Our estimates allow for proving finite element approximation results on Riemannian manifolds. Previous work in this area has restricted largely to hypersurfaces in  $\mathbb{R}^{n+1}$  and submanifolds of Euclidean spaces (e.g., [DE13, BCOS01, HP11]). Often these works use embedded polyhedra to construct finite elements, together with shortest distance maps that map Euclidean simplices to the manifold by assigning each point on the polyhedron to the closest point on the manifold, an approach dating back to [Dzi88]. In many cases, the barycentric finite elements and barycentric coordinates described here can be used in place of this construction.

As an example, consider the the Poisson equation on a closed Riemannian manifold, i.e.,

$$\Delta_g u = f, \quad (5)$$

where  $\Delta_g$  is the Riemannian Laplacian. Equipped with some space of Riemannian finite elements  $V_h$  to be determined, consider the Galerkin approximation  $u_h \in V_h$  that solves

$$\int_M g(du_h, dv) \, \text{dvol}_g = \int_M f v \, \text{dvol}_g \quad \text{for all } v \in V_h, \quad (6)$$

where  $\text{dvol}_g$  is the volume form for the metric  $g$ . Then for the solutions  $u$  and  $u_h$  of Equations (5) and (6), respectively, the following estimates are analogues of the usual method of proving convergence of a Galerkin approximation: (a) Céa's Lemma:

$$\|u - u_h\|_{H^1} \leq c \inf_{v \in V_h} \|u - v\|_{H^1}.$$

and (b) an interpolation estimate:

$$\inf_{v \in V_h} \|u - v\|_{H^1} \leq ch \|u\|_{H^2} \quad \text{for all } u \in H^2.$$

Given these, one can derive the estimates (in case the Poisson problem is  $H^2$ -regular)

$$\|u - u_h\|_{H^1} \leq c_1 \inf_{v \in V_h} \|u - v\|_{H^1} \leq c_2 h \|u\|_{H^2} \leq c_3 h \|f\|_{L^2},$$

proving convergence of  $u_h$  to  $u$ .

The piecewise flat manifolds arising from our construction using Karcher simplices naturally provide piecewise linear finite element spaces. While Céa's lemma is relatively straightforward in this setting, the interpolation lemma requires control over second derivatives of the map  $x$  on each simplex. Theorem 1 provides the requisite estimate.

Similar to this example, the results of our Theorems 20 and 22 pave a natural way to carry over the existing Finite Element approaches for heat flow [DE13], Hodge decomposition [PP00], a weak shape operator [HPW06] and minimal submanifolds [PP93] from embedded surfaces to arbitrary smooth manifolds with an appropriate triangulation. These applications have been detailed in [Dey13].

## 1.4 Related approaches using the Karcher Mean

Usage of the barycentric coordinates is mostly split into three non-intersecting communities from statistics, geometry and numerics.

The Karcher mean has been a standard tool in statistical environments, where interpolation in nonlinear matrix spaces is very frequent, for a relatively long time [Moa05]. We only refer to overview articles [Afs11, Ken13] and the references therein. To these researchers, Karcher means are an averaging method for given data points in a manifold.

Coming from the purely geometric perspective of triangulations, Wintraecken *et al.* have dealt with the distortion of barycentric coordinates using Topogonov’s angle theorem, arriving at a similar result for  $x^*g - g^e$  as our eqn. (3) [DVW15], but with no analogue for eqn. 4. Their focus now goes into the direction of well-definedness and continuity of the coordinates across simplex boundaries [DVW16].

In the theory of finite elements, the group around Grohs and Sander considers problems about maps *into* manifolds, whereas our results have their primary use when mapping from a manifold into, say, the real numbers. From this perspective, barycentric coordinates are an interpolation procedure for vertices which are themselves subject to an optimization problem [San12, GHS13]. Important questions from this point of view include higher-order interpolation [San13], test vector fields [San16], and the rigorous treatment of other problems such as nonlinear elliptical energies or function-space gradient flows [Har15].

The definition of a Karcher simplex would carry over to length spaces in a natural way, but we are not aware of any work in this direction yet.

## 1.5 Structure of the paper

The main part of this article will be structured in three parts: First an overview of flat *simplex* metrics and barycentric coordinates on them, as we rely on several facts for them that are seldom found together in one reference. This also includes the tangent space simplices which we need for the last main part. Next comes the *Riemannian* geometry part, with a more thorough introduction of the main construction than we could give in the introduction. In the final part we combine the Euclidean simplex estimates with smooth Riemannian geometry estimates to bound the distortion of the barycentric coordinate map to the Karcher simplex.

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## 2 Flat Metrics and Barycentric Coordinates

Before considering general Riemannian metrics, we summarize a few facts about Euclidean simplices. The metric of a Euclidean simplex is uniquely determined

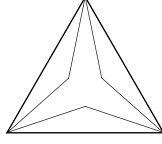


Figure 1: One equilateral triangle and three very thin triangles cannot form the boundary of a Euclidean tetrahedron.

by the length of its edges. Not every system of edge lengths, however, gives rise to a Euclidean simplex – even if the triangle inequality is satisfied. Consider, for example, the situation in Figure 1. In this section we discuss the existence of flat metrics from a given set of edge lengths.

## 2.1 Parametrizations of Euclidean simplices

Before defining general simplices, we introduce two useful special ones.

**Definition 4.** Let  $e_0, \dots, e_n$  be the standard basis of  $\mathbb{R}^{n+1}$ . The *standard simplex*  $\Delta \subset \mathbb{R}^{n+1}$  is the convex hull of the points  $\{e_0, \dots, e_n\}$ . The *unit simplex*  $D \subset \mathbb{R}^n$  is the convex hull of the points  $\{0, e_1, \dots, e_n\}$ .

A general simplex may then be parametrized either over the standard simplex  $\Delta$  or the unit simplex  $D$ .

**Definition 5.** A  $n$ -dimensional Euclidean simplex  $s$  is the convex hull of  $n+1$  points  $p_0, \dots, p_n \in \mathbb{R}^m$ . We define the barycentric map

$$x : \Delta \rightarrow s, \quad x(\lambda) = \sum_{i=0}^n \lambda^i p_i,$$

which gives a parametrization of  $s$  over the standard simplex  $\Delta$ . We may also use a parametrization

$$y : D \rightarrow s, \quad y(u) = Au + p_0,$$

where  $A$  is the matrix with columns  $p_i - p_0$ .

The mapping  $y$  is usually called “mapping onto the reference element,” see, e.g., [BS08, Thm. 4.4.4].

## 2.2 Riemannian metrics on Euclidean simplices

A Euclidean simplex  $s$  inherits a Riemannian metric from its ambient space. We may then pull back this metric to either  $\Delta$  or  $D$ .

First, consider the case of  $\Delta$ . Let  $v, w \in T_\lambda \Delta$ . Then the pullback of the Riemannian metric from  $s$  onto  $\Delta$  is given by

$$\left\langle \sum_{i=0}^n v^i p_i, \sum_{j=0}^n w^j p_j \right\rangle_{\mathbb{R}^m} = \sum_{i,j=0}^n E_{ij} v^i w^j$$

with

$$E_{ij} = -\frac{1}{2}\|p_i - p_j\|_{\mathbb{R}^m}^2$$

for  $i, j = 0, \dots, n$  since  $\sum v^i = 0$  and  $\sum w^i = 0$ . (See [Fie11, Thm. 1.2.9].) The metric is not determined in direction  $e := (1, \dots, 1) \perp_{\mathbb{R}^{n+1}} T_\lambda \Delta$ , so every  $\tilde{E}_{ij} = E_{ij} + \rho$ ,  $\rho \in \mathbb{R}$ , gives the same metric on  $\Delta$ . Since  $s$  is a Euclidean simplex, the symmetric matrix of negative squared edge lengths,  $(E_{ij})$ , yields a positive definite bilinear form when restricted to  $T_\lambda \Delta$ . Conversely, if a system of prescribed “edge lengths”  $\tilde{\ell}_{ij}$  is given such that the matrix with entries  $-\tilde{\ell}_{ij}^2$  yields a positive definite bilinear form on  $T_\lambda \Delta$ , then there exists a Euclidean simplex with edge lengths  $\tilde{\ell}_{ij}$  (see [Fie11, thm. 1.2.4]). The Riemannian metric determined by  $E$  will be generally referred to as  $g^e$ .

Now consider the case of  $D$ . The pullback of the Euclidean metric on  $s$  to the unit simplex  $D$  is given by the bilinear form  $g_{ij}^{\text{eucl}} = \langle p_i - p_0, p_j - p_0 \rangle_{\mathbb{R}^m}$ ,  $i, j = 1, \dots, n$ . Hence

$$g_{ij}^{\text{eucl}} = E_{ij} - E_{0i} - E_{0j} , \quad (7)$$

and the existence criterion for a Euclidean simplex with prescribed edge lengths is equivalent to positive definiteness of  $g_{ij}^{\text{eucl}}$  over  $\mathbb{R}^n$ , see [Fie11, Thm. 1.2.7] or [DW87]. Note that (7) is not true for  $\tilde{E}_{ij}$  as above, but relies on the specific choice that has been taken for  $E_{ij}$  in the direction orthogonal to  $T_\lambda \Delta$ . Since  $\det g_{ij}^{\text{eucl}} = (n! \text{vol } s)^2$ , a  $(\vartheta, h)$ -full Euclidean simplex satisfies  $\det g_{ij}^{\text{eucl}} \geq \vartheta^2 h^{2n}$ . We can use this fact to estimate the eigenvalues of  $g^{\text{eucl}}$ .

**Lemma 3.** *Let  $p_0, \dots, p_n \in \mathbb{R}^m$  be the vertices of a  $(\vartheta, h)$ -full Euclidean  $n$ -simplex, and let  $g_{ij}^{\text{eucl}} = \langle p_i - p_0, p_j - p_0 \rangle_{\mathbb{R}^m}$  denote the pull-back of its metric to the unit simplex  $D$ . Then the eigenvalues  $\lambda_k$  of  $g^{\text{eucl}}$  satisfy*

$$\vartheta h n^{1-n} \leq \sqrt{\lambda_k} \leq h n.$$

*Proof.* Using the matrix  $A$  from Definition 5, notice that  $g^{\text{eucl}} = A^t A$ . Hence the eigenvalues of  $g^{\text{eucl}}$  are the squared singular values of the matrix  $A$ . For any  $n$ -simplex  $s$ , the radius  $r$  of its insphere satisfies  $\text{vol}_n(s) = \frac{r}{n} \text{vol}_{n-1}(\partial s)$ . As  $D$  has volume  $\frac{1}{n!}$  and  $\partial D$  has volume  $\frac{n+\sqrt{n}}{(n-1)!}$ , this gives

$$r = \frac{1}{n + \sqrt{n}} \geq \frac{1}{2n} .$$

This means that any vector  $v \in TD$  with length  $\frac{1}{n} \leq 2r$  can be represented as  $p - q$  with points  $p, q \in D$ . Its image in  $s$  is  $Ap - Aq$ , which must be shorter than the diameter of  $s$ . Since the diameter of a Euclidean simplex is the length of its longest edge, it follows that  $\|A\| \leq nh$ , which implies that  $\lambda_{\max} \leq (nh)^2$ . On the other hand,

$$\lambda_{\min} (nh)^{2n-2} \geq \lambda_{\min} \lambda_{\max}^{n-1} \geq \det g^{\text{eucl}} \geq \vartheta^2 h^{2n} ,$$

which proves the claim.  $\square$



**Remark 6.** One can also express the volume of  $s$  using the matrix  $E$  representing the metric of  $s$  parametrized over the standard simplex  $\Delta$ . Let  $e = (1, \dots, 1) \in \mathbb{R}^{n+1}$ . The volume of  $s$  is  $\frac{2}{n!}(-\det M_+)^{1/2}$ , where

$$M_+ = \begin{pmatrix} 0 & -\frac{1}{2}e^t \\ -\frac{1}{2}e & E \end{pmatrix} \in \mathbb{R}^{(n+2) \times (n+2)}$$

is  $-\frac{1}{2}$  times the *Cayley-Menger* matrix [Fie11, Rem. 1.4.4] (a more classical reference is [Blu52, Thm. 40.1]).

The previous result allows the following estimate.

**Proposition 4.** *Let  $v \in T_\lambda \Delta$  and let  $Y_0, \dots, Y_n$  be vectors at a point on a Riemannian manifold  $(M, g)$ . If  $g^e$  is a Euclidean metric on  $\Delta$  making  $\Delta$  a  $(\vartheta, h)$ -full simplex, then we have the following estimate:*

$$\left| \sum_{i=0}^n v^i Y_i \right|_g \leq \frac{n^n}{\vartheta h} |v|_{g^e} \sum_{i=0}^n |Y_i|_g.$$

*Proof.* The key fact is that we can pull back to the unit simplex  $D$  with pulled back metric  $g^{\text{eucl}}$  and estimate with Lemma 3:

$$\begin{aligned} \left| \sum_{i=0}^n v^i Y_i \right|_g &= \left| \sum_{i=1}^n v^i (Y_i - Y_0) \right|_g \\ &\leq \sqrt{\sum_{i=1}^n |v^i|^2} \sqrt{\sum_{i=1}^n |Y_i - Y_0|_g^2} \\ &\leq \frac{n^n}{\vartheta h} |v|_{g^e} \sum_{i=0}^n |Y_i|_g. \end{aligned}$$

□

As a straightforward consequence of the polarization identity, estimates for symmetric bilinear forms can be derived from estimates of the associated quadratic forms:

**Lemma 5.** *Let  $T$  be a symmetric bilinear form on a vector space  $V$ , and let  $g$  be an inner product on  $V$  (i.e., a symmetric and positive definite bilinear form). Suppose that  $|T(v, v)| \leq C|v|_g^2$  for all vectors  $v$  in  $V$ . Then for all vectors  $v, w \in V$ ,  $|T(v, w)| \leq C|v|_g|w|_g$ .*

The following is a standard estimate.

**Lemma 6.** *Let  $g$  and  $\bar{g}$  be inner products on  $\mathbb{R}^n$  such that all eigenvalues of  $g$  (with respect to the Euclidean inner product) are larger than  $\lambda_{\min} > 0$  and  $|g_{ij} - \bar{g}_{ij}| \leq \varepsilon n^{-1} \lambda_{\min}$ . Then  $|(g - \bar{g})(v, v)| \leq \varepsilon |v|_g^2$ .*

We now apply this lemma to Euclidean metrics arising from simplices.

**Proposition 7.** *There is a constant  $\alpha = \alpha(n)$  with the following property: If  $\ell_{ij}$  are the edge lengths of a Euclidean  $n$ -simplex that defines a  $(\vartheta, h)$ -full metric  $g$  on  $\Delta$ , and  $\bar{\ell}_{ij}$  define a second system of lengths (not a priori assumed to define a simplex) with  $|\ell_{ij} - \bar{\ell}_{ij}| \leq \alpha \varepsilon \vartheta^2 \ell_{ij}$ , where  $\varepsilon < \frac{1}{2}$ , then there is a Euclidean  $n$ -simplex  $\bar{s}$  with edge lengths  $\bar{\ell}_{ij}$ , and its Euclidean metric  $\bar{g}$  over  $\Delta$  satisfies  $|(g - \bar{g})(v, v)| \leq \varepsilon |v|_g^2$  for every  $v \in T\Delta$ .*

*Proof.* Let  $x, y > 0$  be real, such that  $|x - y| \leq \delta x$ , for some  $\delta < 1$ , then  $|x^2 - y^2| \leq 3\delta x^2$ . By assumption and since  $\vartheta \leq 1$ , we hence have  $|\ell_{ij}^2 - \bar{\ell}_{ij}^2| \leq 3\varepsilon \alpha \vartheta^2 \ell_{ij}^2$  whenever  $\alpha \leq 2$ . Therefore,  $|E_{ij} - \bar{E}_{ij}| \leq \frac{3}{2}\varepsilon \alpha \vartheta^2 h^2$ . Working over the unit simplex and using (7) to define  $g^{\text{eucl}}$  and  $\bar{g}^{\text{eucl}}$  we obtain that  $|g_{ij}^{\text{eucl}} - \bar{g}_{ij}^{\text{eucl}}| \leq \frac{9}{2}\varepsilon \alpha \vartheta^2 h^2$ .

Let  $\lambda_{\min}$  be the smallest eigenvalue of  $g^{\text{eucl}}$ . If  $\alpha$  is chosen so small that  $\frac{9}{2}\alpha n \leq n^{2-2n}$ , then by Lemma 3 we have that  $\frac{9}{2}\alpha n \vartheta^2 h^2 \leq \lambda_{\min}$ . By Lemma 6, we get  $|(g - \bar{g})(v, v)| \leq \varepsilon |v|_g^2$ . In particular,  $\bar{g}$  is positive definite since  $\varepsilon < \frac{1}{2}$ .  $\square$

### 2.3 Euclidean simplices from geodesic simplices

Let  $(M, g)$  be a complete Riemannian manifold, and let  $p \in M$ . Then Proposition 7 can be used to compare the flat metric on a Euclidean simplex whose edge lengths are defined by geodesic distances  $\mathbf{d}(\exp_p v, \exp_p w)$  to the flat metric on the Euclidean simplex whose edge lengths are defined by distances  $|v - w|_{g_p}$  in the tangent space at  $p$ . In order to use the conclusions of Proposition 7, we require estimates on the distortion induced by the exponential map. In this section, we provide these estimates.

Before discussing distortion induced by the exponential map, we compare lengths of geodesics for different metrics.

**Proposition 8.** *Let  $g$  and  $g'$  be Riemannian metrics on an open set  $U \subset \mathbb{R}^n$  and let  $\gamma$  and  $\gamma'$  be minimizing geodesics for  $g$  and  $g'$  contained entirely in  $U$  such that  $\gamma(0) = \gamma'(0)$  and  $\gamma(1) = \gamma'(1)$ . If there exists  $0 \leq \alpha < 1$  such that*

$$||v|_g - |v|_{g'}| \leq \alpha |v|_g$$

*for all  $v \in TU$ , then the lengths of  $\gamma$  and  $\gamma'$  satisfy*

$$|l_g(\gamma) - l_{g'}(\gamma')| \leq \alpha l_g(\gamma) .$$

*Proof.* We first note that

$$(1 - \alpha)|v|_g \leq |v|_{g'} .$$

Since  $\gamma$  is minimizing for  $g$ , we have

$$\begin{aligned} (1 - \alpha)l_{g'}(\gamma') &= (1 - \alpha) \int_0^1 |\dot{\gamma}'|_g dt + (1 - \alpha) \int_0^1 (|\dot{\gamma}'|_{g'} - |\dot{\gamma}'|_g) dt \\ &\geq (1 - \alpha)l_g(\gamma') - \alpha(1 - \alpha) \int_0^1 |\dot{\gamma}'|_g dt \\ &\geq (1 - \alpha)l_g(\gamma) - \alpha l_{g'}(\gamma') . \end{aligned}$$

It follows that

$$l_g(\gamma) - l_{g'}(\gamma') \leq \alpha l_g(\gamma).$$

Similarly, since  $\gamma'$  is minimizing for  $g'$ , we have

$$\begin{aligned} l_g(\gamma) &= \int_0^1 |\dot{\gamma}|_{g'} dt + \int_0^1 (|\dot{\gamma}|_g - |\dot{\gamma}|_{g'}) dt \\ &\geq l_{g'}(\gamma) - \alpha l_g(\gamma) \\ &\geq l_{g'}(\gamma') - \alpha l_g(\gamma), \end{aligned}$$

so

$$l_{g'}(\gamma') - l_g(\gamma) \leq \alpha l_g(\gamma).$$

□

The main comparison result of this section is the following.

**Proposition 9.** *Let  $(M, g)$  be a complete Riemannian manifold. Let  $p \in M$ , and let  $U \subset T_p M$  be a ball of radius  $r$  centered at  $0 \in T_p M$  such that  $\exp_p(U)$  is geodesically convex. Pull back the metric  $g$  to  $U$  via  $\exp_p$ .*

*Suppose  $c : [0, \tau] \rightarrow U$  is a geodesic (with respect to the pulled back metric) and  $b : [0, \tau] \rightarrow U$  is a straight line with same endpoints  $b(0) = c(0)$  and  $b(\tau) = c(\tau)$ . There are constants  $\alpha = \alpha(C_0, n)$  and  $\beta = \beta(n)$  such that if  $r < \alpha$ , then the  $g$ -lengths of  $c$  and the Euclidean length of  $b$  satisfy*

$$|l_g(c) - l_{\text{eucl}}(b)| \leq \beta C_0 r^2 l_g(c).$$

*Proof.* For a geodesic ball with radius  $r < 1$ , there is a constant  $\beta = \beta(n)$  with

$$||v|_g - |v|_{\text{eucl}}| \leq \beta C_0 r^2 |v|_g,$$

for all  $v \in TU$  [Kau76]. If we suppose  $r < \alpha$  with  $\alpha < 1$  so small that

$$\beta C_0 \alpha^2 < 1,$$

then the proposition follows from Proposition 8. □

The following is the main result we will use in Section 4.

**Corollary 10.** *Let  $(M, g)$  be a complete Riemannian manifold. Let  $a \in M$  and suppose that  $p_0, \dots, p_k$  are points in a convex geodesic ball centered at  $a$  such that the lengths  $\ell_{ij} = \mathbf{d}(p_i, p_j)$  form a Euclidean simplex and let  $g^e$  denote the induced metric on  $\Delta$ . Let  $X_i \in T_a M$  be defined such that  $\exp_a X_i = p_i$  and let  $\sigma : \Delta \rightarrow T_a M$  denote the barycentric coordinates to the simplex determined by the  $X_i$ .*

*There are  $\alpha = \alpha(n)$  and  $\beta = \beta(n)$  such that if  $g^e$  is a  $(\vartheta, h)$ -full metric with  $h < \alpha$  then*

$$||v|_{g^e}^2 - |\sigma(v)|_g^2| \leq \beta C_0 \vartheta^{-2} h^2 |v|_{g^e}^2$$

*for every  $v \in T\Delta$ . It follows that there is an  $\alpha' = \alpha'(n, C_0, \vartheta)$  so that if furthermore  $h < \alpha'$ , then*

$$||v|_{g^e}^2 - |\sigma(v)|_g^2| \leq 2\beta C_0 \vartheta^{-2} h^2 |\sigma(v)|_g^2.$$

Note that  $|\sigma(v)|_g$  is equal to the length of the vector  $v$  measured in the Euclidean metric  $g^e$  on  $\Delta$  determined by the lengths  $\bar{\ell}_{ij} = |X_i - X_j|$ .

*Proof.* The fact that there is a  $\gamma = \gamma(n)$  such that for each corresponding side,

$$|\ell_{ij} - \bar{\ell}_{ij}| \leq \gamma C_0 r^2 \ell_{ij}$$

follows from Proposition 9. Taking

$$\varepsilon = \frac{\gamma}{\alpha} C_0 \vartheta^{-2} h^2$$

in Proposition 7 (using the  $\alpha$  from that proposition) we get that

$$||v|_{g^e}^2 - |\sigma(v)|_g^2| \leq \frac{\gamma}{\alpha} C_0 \vartheta^{-2} h^2 |v|_{g^e}^2.$$

The last statement follows from the fact that

$$|v|_{g^e}^2 \leq ||v|_{g^e}^2 - |\sigma(v)|_g^2| + |\sigma(v)|_g^2.$$

so we need to take  $\alpha'$  small enough so that  $\beta C_0 \vartheta^{-2} (\alpha')^2 < 1/2$ .  $\square$

One last fact that will be important in the sequel.

**Proposition 11.** *In Proposition 9 and Corollary 10, we may replace the convexity assumption by making the constants  $\alpha$  depend also on the injectivity radius  $\iota$ .*

*Proof.* This is the well-known fact that every point has a convex geodesic ball around it, and the radius of that ball can be estimated by the curvature bound  $C_0$  and the injectivity radius  $\iota$ .  $\square$

### 3 Barycentric coordinates from center of mass

#### 3.1 Definition of the barycentric coordinate map

Let  $(M, g)$  be a complete  $m$ -dimensional Riemannian manifold,  $m > 1$ , and  $\Delta$  be the  $n$ -dimensional standard simplex  $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda^i \geq 0, \sum \lambda^i = 1\}$ . For points  $p_0, \dots, p_n \in M$ , consider the function

$$E : M \times \Delta \rightarrow \mathbb{R}$$

given by

$$E(a, \lambda) = \lambda^0 \mathbf{d}^2(a, p_0) + \dots + \lambda^n \mathbf{d}^2(a, p_n).$$

The minimizer of  $E(\cdot, \lambda)$ , if it exists, is called the **center of mass** for the measure  $\sum \lambda^i \delta_{p_i}$ , where  $\delta_p$  is the Dirac delta point mass at the point  $p$ . It is especially useful to see that the center of mass can be formulated as the zero of a function.

**Proposition 12** ([Kar77]). *Local minimizers of  $E(\cdot, \lambda)$  for fixed  $\lambda$  are zeroes of the section  $F : M \times \Delta \rightarrow TM$  given by*

$$F(a, \lambda) = \lambda^0 X_0|_a + \cdots + \lambda^n X_n|_a, \quad \text{where } X_i = \frac{1}{2} \text{grad } \mathbf{d}^2(\cdot, p_i). \quad (8)$$

Given that the center of mass is just a solution to  $F = 0$ , where  $F$  is given by Equation 8, one can apply the implicit function theorem to get a solution. One caution is that in order to apply the implicit function theorem, one needs to trivialize the bundle  $TM$ , and hence we must choose a connection to do the trivialization. It is natural to use the Riemannian connection. In later computations, we will see the connection appear in formulas for differentials of mappings (not just derivatives of forms and vector fields), and this is because the definition of the mapping  $F$  requires this trivialization.

The existence of the center of mass is hard to prove in general, cf. [San13]. For sufficiently small balls, however, there is a unique minimizer. The following proposition is a generalization of one in [Kar77] due to Kendall [Ken90] (see also [Afs11]).

**Proposition 13.** *If the points  $p_i$  lie in a ball whose radius is less than half the convexity radius, then  $E(\cdot, \lambda)$  has a unique minimizer.*

Since we will always want to consider unique centers of mass, we make the following assumption.

**Assumption.** From the rest of this paper, we only consider  $p_0, \dots, p_n$  that lie in a ball  $B$  whose radius is less than half the convexity radius. Note that, as in Proposition 11, if we choose ball to have small enough radius, where “small enough” depends on the curvature bound  $C_0$  and the injectivity radius  $\iota$ , this is true (the original bound in [Kar77] is of this form). Hence this extra assumption is superfluous if we require our future antecedents of the form  $h < \alpha$  to have  $\alpha$  depend on  $\iota$  as well as  $C_0$ .

We are now able to define the barycentric mapping.

**Definition 7.** For a given  $\lambda \in \Delta$ , let  $x(\lambda)$  be the minimizer of  $E(\cdot, \lambda)$  in  $B$ . We call  $x$  the **barycentric mapping** with respect to vertices  $p_1, \dots, p_n$ . Its image in  $M$  is called the corresponding **Karcher simplex**.

**Remark 8.** **(a)** In case  $M$  is the Euclidean space,  $x$  is just the canonical parametrisation  $\lambda \mapsto \sum \lambda^i p_i$ , because  $\mathbf{d}^2(p, q) = |q - p|^2$  and  $X_i|_q = q - p_i$ .

**(b)** For  $\lambda^i = 0$ , the value  $x(\lambda)$  is independent of  $p_i$ . So the facets of the standard simplex are mapped to “Karcher subsimplices” which only depend on the vertices of the subsimplex.

**(c)** Because  $x$  is continuous, the Karcher subsimplices form the boundary of a Karcher simplex:  $\partial(x(\Delta)) = x(\partial\Delta)$ .

The barycentric mapping behaves well with respect to submanifolds.

**Proposition 14** (geodesic submanifolds). *Let  $e_i$  be the  $i$ -th Euclidean basis vector of  $\mathbb{R}^{n+1}$ . Then  $x(te_j + (1-t)e_i) = \gamma(t)$ , where  $\gamma$  is the unique shortest geodesic with  $\gamma(0) = p_i$  and  $\gamma(1) = p_j$ . If all  $p_i$  lie in a common totally geodesic submanifold  $N \subset M$ , then so does  $x(\lambda)$  for arbitrary  $\lambda \in \Delta$ .*

*Proof.* The first claim is clear. For the second, we will show that the center of mass restricted to  $N$  is also the center of mass on  $M$  since  $N$  is totally geodesic. Let  $k_i := \frac{1}{2} \mathbf{d}^2(\cdot, p_i)$ . As  $E$  is convex on  $B$ , there is a unique minimizer  $a$  of  $E(\lambda, \cdot)|_N$ , so there are coefficients  $\lambda^0, \dots, \lambda^n$  with

$$\lambda^0 \text{grad}(k_0|_N) + \dots + \lambda^n \text{grad}(k_n|_N) = 0.$$

As  $N$  is totally geodesic in  $M$ , it follows that  $\text{grad}(k_i|_N) = (\text{grad } k_i)|_N = X_i|_N$ . Hence  $\lambda^0 X_0 + \dots + \lambda^n X_n = 0$  at the point  $a \in N$ . Thus  $a = x(\lambda)$ .  $\square$

**Remark 9.** In general,  $s$  is the convex hull of the  $p_i$  (the smallest convex set containing all  $p_i$ ) if and only if all subsimplices of  $s$  are totally geodesic. In fact, if a Karcher triangle is not totally geodesic, the geodesics connecting any points on the geodesic edges will not be contained in the triangle.

**Bibliographic note.** We already remarked that one can consider  $\lambda$  as a point measure on  $M$  that assigns the mass  $\lambda^i$  to the point  $p_i$ . For a general probability measure  $\mu$  on  $M$ , [Kar77] speaks of the minimizer of

$$E_\mu(a) := \int_M \mathbf{d}^2(a, p) \, d\mu(p)$$

as “Riemannian center of mass”, but the subsequent literature has mostly called it the “Karcher mean” with respect to the measure  $\mu$ , probably initiated by Kendall in [Ken90] (unfortunately, Karcher himself is not very happy with the naming, see [Kar14]). The concept seems to go back to Cartan (see the historic overview in [Afs11]), but had not been used by others until the work of [GK73].

Karcher himself used the center of mass to retrace the standard mollification procedure of Gauss kernel convolution in the case of functions that map into a manifold. Considering the center of mass as a function from an interesting finite-dimensional space of measures into  $M$ , as we use it, has been done by [Rus10]. For other recent applications, see the discussion in sec. 1.4.

### 3.2 Derivatives of the barycentric coordinate map

Since the next proposition uses pullback bundles, we take this opportunity to recall the calculus of pullback vector bundles. Recall that given a smooth vector bundle

$$\pi : E \rightarrow M$$

and smooth map

$$\varphi : N \rightarrow M,$$

one can define the pullback bundle

$$\varphi^*\pi : \tilde{E} \rightarrow N$$

with

$$\tilde{E} = \bigsqcup_{x \in N} E_{\varphi(x)},$$

where  $\sqcup$  represents the disjoint union. (We will always be considering injective maps, so the disjoint union may be replaced with the union.) For the tangent bundle of  $M$ , we use the notation  $\varphi^*TM$ . If  $V$  is a vector field on  $M$ , we will denote corresponding sections  $\varphi^*TM \rightarrow N$  by  $\tilde{V}|_p = V|_{\varphi(p)}$ . We will not stress the discrepancy between the section  $\tilde{V}$  evaluated at  $p$  and the section  $V$  evaluated at  $\varphi(p)$ , even though the latter requires a (more general) vector field on  $M$ . If we have a smooth map  $\varphi$ , the differential  $d\varphi$  is a section of the bundle

$$T^*N \otimes \varphi^*TM \rightarrow N$$

(the dual-space star in  $T^*N$  is not to be confused with the pull-back star in  $\varphi^*TM$ ). Given Riemannian metrics  $g^N$  and  $g^M$  on  $N$  and  $M$  with Riemannian connections  $\nabla^N$  and  $\nabla^M$ , the Hessian  $\nabla d\varphi$  is a section of the bundle

$$T^*N \otimes T^*N \otimes \varphi^*TM \rightarrow N,$$

given by differentiating the section  $d\varphi$  using the induced connection on the bundle  $T^*N \otimes \varphi^*TM \rightarrow N$ .

In particular, considering the map  $x : (\Delta, g^e) \rightarrow (M, g)$ , we get the following derivatives:

⟨a⟩ The section  $dx$  of the bundle  $T^*\Delta \otimes x^*TM \rightarrow \Delta$ . We denote  $V = dx(v)$ .

⟨b⟩ The section  $\nabla dx$  of the bundle  $T^*\Delta \otimes T^*\Delta \otimes x^*TM \rightarrow \Delta$  given by

$$\nabla dx(v, w) = \nabla_{dx(v)}dx(w) - dx(\nabla_v w) = \nabla_V W - dx(\nabla_v w), \quad (9)$$

where the first connection on the right is the Riemannian connection for  $g$  and the second connection is the Riemannian connection for  $g^e$ .

The following is a direct consequence of Proposition 12.

**Proposition 15.** *The map*

$$G : \Delta \rightarrow x^*TM$$

*given by*

$$\begin{aligned} G(\lambda) &= F(x(\lambda), \lambda) \\ &= \sum \lambda^i X_i|_{x(\lambda)} \end{aligned}$$

*is the zero section.*

It follows that  $dG$  is the zero section of the bundle  $T^*\Delta \otimes x^*TM \rightarrow \Delta$ , and hence

$$0 = dG(v) = \sum v^i X_i + \sum \lambda^i \nabla_{dx(v)} X_i. \quad (10)$$

We will use this and also its derivative to calculate  $dx$  and  $\nabla dx$ . It will be useful to name the two parts to the above formula.

**Definition 10** (cf. Corollary 10). Define the section  $\sigma$  of the bundle  $T^*\Delta \otimes x^*TM \rightarrow \Delta$  to be

$$\sigma(v) = -\sum v^i X_i$$

and define the section  $A$  of the bundle  $(x^*TM)^* \otimes x^*TM \rightarrow \Delta$  to be

$$A(V) = \sum \lambda^i \nabla_V X_i,$$

where each of these sections is evaluated at  $\lambda \in \Delta$ .

These are defined so that

$$dG(v) = -\sigma(v) + A(dx(v)).$$

The relationship of  $\sigma$  and  $A$  with  $dx$  and  $\nabla dx$  is given by the following.

**Proposition 16.** *Let  $v$  and  $w$  be vector fields on  $\Delta$  and let  $V = dx(v)$  and  $W = dx(w)$  be the corresponding sections of  $x^*TM \rightarrow \Delta$ . The differential  $dx$  and Hessian  $\nabla dx$  satisfy*

$$A(dx(v)) = \sigma(v),$$

and

$$A(\nabla dx(v, w)) = -\left(\sum w^i \nabla_V X_i + \sum v^i \nabla_W X_i + \sum \lambda^i \nabla_{W,V}^2 X_i\right),$$

evaluated at a point  $\lambda \in \Delta$ .

*Proof.* These will both follow from differentiating the identity from Proposition 15; the first claim is already treated by (10). We take the next derivative using the covariant derivative of  $dG$ , which is a section of  $T^*\Delta \otimes x^*TM \rightarrow \Delta$ , using the connection  $\nabla^{e,g}$  determined by the Riemannian connections of  $g^e$  and  $g$ . To simplify the writing, we use  $\nabla$  for each connection, with the understanding that the context will make it clear whether the connection needs to be computed with respect to  $\nabla^{e,g}$ ,  $\nabla^e$ , or  $\nabla^g$  (which is primarily used on the pullback connection



as  $x^*TM \rightarrow \Delta$ ).

$$\begin{aligned}
\nabla dG(w, v) &= \nabla_w dG(v) - dG(\nabla_w v) \\
&= \sum w(v^i)X_i + \sum v^i \nabla_W X_i + \sum w^i \nabla_V X_i \\
&\quad + \sum \lambda^i \nabla_W \nabla_V X_i - \sum (\nabla_w v)^i X_i - \sum \lambda^i \nabla_{dx(\nabla_w v)} X_i \\
&= \sum v^i \nabla_W X_i + \sum w^i \nabla_V X_i + \sum \lambda^i \nabla_{W,V}^2 X_i \\
&\quad + \sum \lambda^i \nabla_{\nabla_W V} X_i - \sum \lambda^i \nabla_{dx(\nabla_w v)} X_i \\
&= \sum v^i \nabla_W X_i + \sum w^i \nabla_V X_i \\
&\quad + \sum \lambda^i \nabla_{W,V}^2 X_i + \sum \lambda^i \nabla_{\nabla dx(w,v)} X_i
\end{aligned}$$

for vector fields  $v$  and  $w$  on  $\Delta$  evaluated at  $\lambda$ . The last equality follows from the definition of  $\nabla dx$  (see Equation 9).  $\square$

## 4 Estimates

In Proposition 7 from Section 2 we saw that if two flat Riemannian metrics on  $\Delta$  assign almost the same lengths to the edges of the standard simplex, the metrics are almost equal. In Proposition 9 we saw that in normal coordinates, geodesic distances and Euclidean distances of the coordinates are almost the same. These will give us the flexibility to give our estimates with respect to the appropriate metrics.

In order to compare  $x^*g$  to  $g^e$ , we add an intermediate step by using  $\sigma$  from Definition 10. For a fixed  $\lambda$ , consider the Euclidean simplex  $\bar{s}$  in the tangent space  $T_{x(\lambda)}M$  with vertices  $X_i|_{x(\lambda)} = (\exp_{x(\lambda)})^{-1}p_i$ . The (classical) barycentric coordinate map for this simplex is, in fact,  $\sigma$ . The simplex  $\bar{s}$  inherits the metric  $g_{x(\lambda)}$  from its ambient space  $T_{x(\lambda)}M$ . We show that  $dx - \sigma$  is small, so that  $x^*g$  and  $\sigma^*g$  almost agree. And by Proposition 9,  $|X_i - X_j|$  is almost  $\mathbf{d}(p_i, p_j)$ , hence Proposition 7 compares the flat metrics  $\sigma^*g$  and  $g^e$ . The relevant estimate is Corollary 10.

We now introduce a notation we will use for estimates.

**Notation 11.** By  $a \lesssim b$ , we mean that there is a constant  $\alpha = \alpha(n)$  with  $a \leq \alpha b$ . We decided not to include the geometric properties in this suppressed constant to make their influences clear. In the following, we will give estimates of the form

$$|A - B| \leq \beta |B|.$$

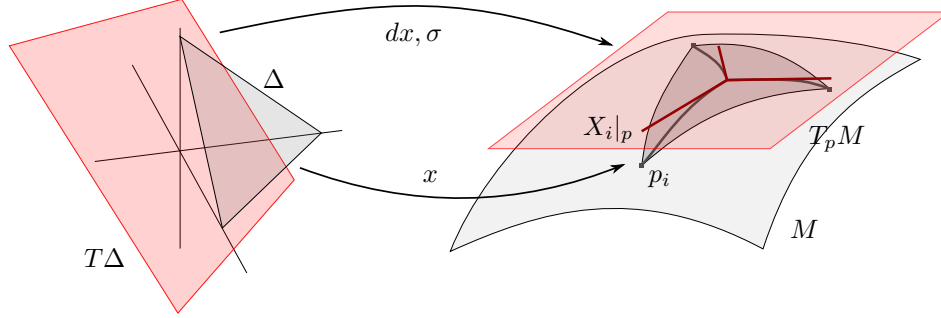
We note that for sufficiently small  $\beta$ , this estimate implies the estimates

$$(1 - \beta)|B| \leq |A| \leq (1 + \beta)|B|$$

since the triangle inequality gives

$$|B| - |A - B| \leq |A| \leq |B| + |A - B|.$$

Figure 2: Notation overview for mappings:  $x$  maps the standard simplex  $\Delta$  into the grey Karcher simplex  $s$ , while  $\sigma$  maps  $\Delta$  onto the convex hull  $\bar{s}$  of the  $X_i|_p$ .



#### 4.1 Preliminary estimates

From Proposition 16, we see that if we can show that  $A$  is invertible and we can bound its operator norm, we can get estimates on  $dx$  and  $\nabla dx$ . First, we will need an estimate for derivatives of the distance function. Recall the notation from Section 1.2 before Theorem 1. The Jacobi field estimate from Corollary 28 in the Appendix shows the following.

**Proposition 17.** *There exists  $\alpha = \alpha(C_0)$  such that if  $h < \alpha$  and  $p_0, \dots, p_n$  are in a convex ball  $B$  of diameter less than  $h$ , then for all  $i = 0, \dots, n$  and for  $V \in T_q M$ , where  $q \in B$ , we have*

$$|\nabla_V X_i - V| \lesssim C_0 h^2 |V|$$

and

$$|\nabla_{V,V}^2 X_i| \lesssim C_{0,1} h |V|^2.$$

We may now estimate  $A$  as an operator. In the next proposition, we use  $\|T\|$  to denote the operator norm of the map  $T$  considered with the metric  $x^*g$ .

**Proposition 18.** *There exists  $\alpha = \alpha(n, C_0)$  such that if  $h < \alpha$ , the section  $A$  of the bundle  $(x^*TM)^* \otimes x^*TM \rightarrow \Delta$  is pointwise invertible, i.e., there is another section  $A^{-1}$  of the same bundle such that for each section  $v$  of the bundle  $x^*TM \rightarrow \Delta$ ,  $A^{-1}(A(v)) = A(A^{-1}(v)) = v$ . Furthermore, there exists  $\beta = \beta(n)$  such that  $A$  satisfies the following estimates:*

$$\|A - \text{id}\| \lesssim C_0 h^2$$

and

$$\begin{aligned} |V|_g &\lesssim |A(V)|_g, \\ |A(V) - V|_g &\lesssim C_0 h^2 |A(V)|_g \end{aligned}$$

for any section  $V$  of  $x^*TM$ .

*Proof.* By Proposition 17, one has

$$|\nabla_V X_i - V| \lesssim C_0 h^2 |V|$$

for all tangent vectors  $V$ , or, in terms of operator norms,

$$\|\nabla X_i - \text{id}\| \lesssim C_0 h^2.$$

As all  $\lambda^i$  are positive and sum up to one, this estimate carries over to

$$\|A - \text{id}\| \lesssim C_0 h^2. \quad (11)$$

It follows that there exists  $\alpha = \alpha(C_0)$  such that if  $h < \alpha$  then  $A$  is invertible.

For the second estimate, we see that

$$|V|_g = |(\text{id} - A)V + A(V)|_g \leq \|\text{id} - A\| |V|_g + |A(V)|_g.$$

Using Equation 11, we get

$$(1 - \beta C_0 h^2) |V|_g \leq |A(V)|_g$$

for some function  $\beta = \beta(n)$  and so if we make  $\alpha$  small enough so  $\beta C_0 \alpha^2 < \frac{1}{2}$  then, we have

$$|V|_g \lesssim |A(V)|_g.$$

Finally, we get

$$|A(V) - V|_g \lesssim \|A - \text{id}\| |V|_g \lesssim \|A - \text{id}\| |A(V)|_g.$$

□

For the rest of this section, let  $p_0, \dots, p_n$  be distinct points inside a convex ball of radius  $h$ . Note that this is not a big assumption since we can just adjust out constants to depend on the injectivity radius as well, as in Proposition 11.

## 4.2 First derivative estimates

We can now estimate  $dx$ .

**Theorem 19.** *There exists  $\alpha = \alpha(n, C_0)$  and  $\alpha' = \alpha'(n, C_0, \vartheta)$  such that if  $h < \alpha$  then*

$$|dx(v) - \sigma(v)| \lesssim C_0 h^2 |\sigma(v)|$$

*and if furthermore  $g^e$  is the metric of a  $(\vartheta, h)$ -full simplex with  $h < \alpha'$ , then*

$$|dx(v) - \sigma(v)| \lesssim C_0 h^2 |v|_{g^e},$$

*for any tangent vector  $v \in T_\lambda \Delta$  at any  $\lambda \in \Delta$ ,*

*Proof.* By Proposition 16,  $A(dx(v)) = \sigma(v)$  and hence the first statement now follows from Proposition 18. Applying Corollary 10 we also get that there is a  $\beta = \beta(n)$  such that

$$|dx(v) - \sigma(v)| \lesssim C_0 h^2 (1 + \beta C_0 \vartheta^{-2} h^2) |v|_{g^e},$$

so we can take  $\alpha' < \alpha$  small enough so that  $\beta C_0 \vartheta^{-2} (\alpha')^2 < 1$  to get the second statement.  $\square$

We can now turn our estimate on  $dx$  into an estimate of the relevant metric tensors.

**Theorem 20.** *There exists  $\alpha = \alpha(n, C_0)$  such that if  $g^e$  is the metric of a  $(\vartheta, h)$ -full simplex with  $h < \alpha$  then for tangent vectors  $v, w \in T_\lambda \Delta$  at any  $\lambda \in \Delta$*

$$|(x^*g - g^e)(v, w)| \lesssim C_0 (1 + \vartheta^{-2}) h^2 |v|_{g^e} |w|_{g^e} \quad (12)$$

and there is an  $\alpha' = \alpha'(n, \vartheta, C_0)$  such that if furthermore  $h < \alpha'$  then

$$|v|_{x^*g}^2 \lesssim |v|_{g^e}^2. \quad (13)$$

*Proof.* Due to Lemma 5, it suffices to show the claim for  $v = w$ . Consider a vector  $v$  at a point  $\lambda \in \Delta$  with image  $a = x(\lambda)$ . We will compare the value of  $|\sigma(v)|_g^2$ . By Theorem 19, we can choose  $\alpha$  small enough so that if  $h \leq \alpha$  then

$$\begin{aligned} ||v|_{x^*g} - |\sigma(v)|_g| &= ||dx(v)|_g - |\sigma(v)|_g| \\ &\leq |dx(v) - \sigma(v)|_g \\ &\lesssim C_0 h^2 |\sigma(v)|_g. \end{aligned}$$

The same is true for the squared norms,

$$||v|_{x^*g}^2 - |\sigma(v)|_g^2| \lesssim C_0 h^2 |\sigma(v)|_g^2 \quad (14)$$

if  $\alpha$  is small enough. Hence we have successfully compared  $x^*g$  to the Euclidean metric  $\sigma^*g$ . By (14),

$$\begin{aligned} |(x^*g - g^e)(v, v)| &\leq ||v|_{x^*g}^2 - |\sigma(v)|_g^2| + ||\sigma(v)|_g^2 - |v|_{g^e}^2| \\ &\lesssim C_0 h^2 |\sigma(v)|_g^2 + ||\sigma(v)|_g^2 - |v|_{g^e}^2| \\ &\lesssim C_0 h^2 |v|_{g^e}^2 + (1 + C_0 h^2) ||\sigma(v)|_g^2 - |v|_{g^e}^2|. \end{aligned}$$

We can now use Corollary 10 to get

$$|(x^*g - g^e)(v, v)| \lesssim (1 + (1 + C_0 h^2) \vartheta^{-2}) C_0 h^2 |v|_{g^e}^2.$$

The first result follows if  $C_0 \alpha^2 < 1$ . The second result follows if we take  $\alpha' < \alpha$  small enough so that  $C_0 (1 + \vartheta^{-2}) (\alpha')^2 < 1$ .  $\square$

### 4.3 Second derivative estimates

We are now ready to prove the estimate for  $\nabla dx$ .

**Theorem 21.** *There exists  $\alpha = \alpha(n, \vartheta, C_0, C_1)$  such that if  $g^e$  is the metric of a  $(\vartheta, h)$ -full simplex with  $h < \alpha$ , then*

$$|\nabla dx(v, w)|_g \lesssim C_{0,1}(1 + \vartheta^{-1})h|v|_{g^e}|w|_{g^e},$$

or, using the operator norm,

$$\|\nabla dx\| \lesssim C_{0,1}(1 + \vartheta^{-1})h$$

when  $x$  is considered as a mapping  $(\Delta, g^e) \rightarrow (M, g)$ .

*Proof.* Throughout the proof we will assume  $h < \alpha$ , and show that we can make  $\alpha$  sufficiently small to prove the estimate. As before, it suffices to show the theorem for  $v = w$ . Let  $V = dx(v)$ . Since the  $v^i$  sum up to zero,

$$\begin{aligned} |v^i \nabla_V X_i|_g &= |v^i \nabla_V X_i - \sum v^i V|_g \lesssim (\vartheta h)^{-1}|v|_{g^e} \sum |\nabla_V X_i - V|_g \\ &\lesssim C_0 \vartheta^{-1} h |v|_{g^e} |V|_g \end{aligned}$$

using Propositions 4 and 17.

Now we use Proposition 16,

$$\begin{aligned} |A(\nabla dx(v, v))|_g &\leq 2|v^i \nabla_V X_i|_g + |\lambda^i \nabla_{dx(v), dx(v)}^2 X_i|_g \\ &\lesssim C_0 \vartheta^{-1} h |v|_{g^e} |V|_g + C_{0,1} h |V|_g^2, \end{aligned}$$

where  $\nabla^2 X_i$  has been estimated by Proposition 17. Since  $|V|_g = |v|_{x^*g}$ , it follows from Theorem 20 that for  $\alpha$  sufficiently small,

$$|A(\nabla dx(v, v))|_g \lesssim C_{0,1} (1 + \vartheta^{-1}) h |v|_{g^e}^2.$$

By Proposition 18,

$$\begin{aligned} |\nabla dx(v, v)|_g &\lesssim |A(\nabla dx(v, v))|_g \\ &\lesssim C_{0,1} (1 + \vartheta^{-1}) h |v|_{g^e}^2. \end{aligned}$$

□

The corresponding estimate for the metric is the following.

**Theorem 22.** *Let the pull-back of the connection on  $M$  be defined as  $dx \nabla_v^{x^*g} w = \nabla_{dx(v)} dx(w)$ . There is a constant  $\alpha = \alpha(n, \vartheta, C_0, C_1)$  such that if  $g^e$  is the metric of a  $(\vartheta, h)$ -full simplex with  $h < \alpha$ , then*

$$|\nabla^e x^* g(u, v, w)| \lesssim C_{0,1} (1 + \vartheta^{-1}) h |u|_{g^e} |v|_{g^e} |w|_{g^e}$$

for tangent vectors  $u, v, w \in T_\lambda \Delta$  at any  $\lambda \in \Delta$ .

We note that the estimate is equivalently a bound on the difference of the Christoffel symbols corresponding to the metrics  $g^e$  and  $x^*g$ , which is a tensor (the difference of connections is a tensor). Since the metric  $g^e$  is constant, this is an estimate for the Christoffel symbols for the metric  $g$  in coordinate  $x$ .

*Proof.* We see that

$$\begin{aligned}\nabla^e x^*g(u, v, w) &= u[g(dx(v), dx(w))] - g(dx(\nabla_u^e v), dx(w)) - g(dx(v), dx(\nabla_u^e w)) \\ &= g(\nabla_{dx(u)} dx(v), dx(w)) + g(dx(v), \nabla_{dx(u)} dx(w)) \\ &\quad - g(dx(\nabla_u^e v), dx(w)) - g(dx(v), dx(\nabla_u^e w)) \\ &= g(\nabla dx(u, v), dx(w)) + g(dx(v), \nabla dx(u, w)),\end{aligned}$$

so Theorem 21 gives the estimate

$$|\nabla^e x^*g(u, v, w)| \lesssim C_{0,1}(1+\vartheta^{-1})h|u|_{g^e}|v|_{g^e}|w|_{x^*g} + C_{0,1}(1+\vartheta^{-1})h|u|_{g^e}|w|_{g^e}|v|_{x^*g}.$$

Applying Theorem 20 gives the result.  $\square$

## 5 Appendix

We promised the reader evidence for Theorem 17, which is given in this appendix.

Throughout this section, we will assume the following notation. Let  $(M, g)$  be a complete  $m$ -dimensional Riemannian manifold with curvature bounds  $\|R\|_\infty \leq C_0$  and  $\|\nabla R\|_\infty \leq C_1$ , where  $R$  denotes the curvature tensor. Suppose  $\gamma : [0; \tau] \rightarrow M$  is the unique arclength-parametrized geodesic with  $\gamma(0) = p$  and  $\gamma(\tau) = q$ , and  $V \in T_q M$ . Let  $s \mapsto \delta(s)$  be a geodesic with  $\delta(0) = \gamma(\tau)$  and  $\frac{d}{ds}\delta(0) = V$ . Define a variation of geodesics by

$$c(s, t) := \exp_p \left( \frac{t}{\tau} (\exp_p)^{-1} \delta(s) \right)$$

so  $c(0, t) = \gamma(t)$ . Then for each small  $s$  and  $t \in [0, \tau]$ ,  $T := \partial_t c$  is tangent to a geodesic curve and  $J(s, t) := \partial_s c$  is a Jacobi field along  $t \mapsto c(s, t)$ . We note that  $|T(0, t)| = 1$  since  $\gamma$  is parametrized by arclength. We denote  $t$  covariant derivatives by a dot, so this means  $\dot{T} = 0$  and  $\dot{J} + R(J, T)T = 0$ .

Recall that we use  $A \lesssim B$  to mean there exists a constant  $\beta = \beta(n)$  such that  $A \leq \beta B$ . Since there is no simplex in this section, and hence no  $n$ , the constant  $\beta$  will be universal. In particular, the constant does not depend on  $m$ , the dimension of  $M$ .

### 5.1 Jacobi Fields with Two Fixed Values

In this section we review some estimates on Jacobi fields.

The first result we need is a form of the Rauch comparison theorem. Here is a reformulation of part of the result [Jos11, Theorem 5.5.1].

**Theorem 23.** *We have the following estimate for  $J(0, t)$ :*

$$g(J(0, t), \dot{J}(0, t)) \geq |J(0, t)|^2 \frac{\sqrt{C_0} \cos \sqrt{C_0} t}{\sin \sqrt{C_0} t}.$$

*In particular,  $|J(0, t)|$  is increasing in  $t$  if  $t < \frac{\pi}{2\sqrt{C_0}}$ .*

*Proof.* The first part is essentially the first part of [Jos11, Theorem 5.5.1]. The second follows because

$$\frac{d}{dt}|J(0, t)| = \frac{g(J(0, t), \dot{J}(0, t))}{|J(0, t)|}$$

and both  $\sin \sqrt{C_0} t$  and  $\cos \sqrt{C_0} t$  are positive if  $t < \frac{\pi}{2\sqrt{C_0}}$ .  $\square$

**Lemma 24.** *Suppose  $\tau < \frac{\pi}{2\sqrt{C_0}}$ . Then*

$$|\tau \dot{J}(0, \tau) - V| \lesssim C_0 \tau^2 |V|$$

*and*

$$|\dot{J}(0, \tau)| \lesssim \frac{1}{\tau} (1 + C_0 \tau^2) |V|.$$

*Proof.* By Theorem 23 we have that  $|J|$  is increasing for all  $t < \tau$ . Now observe  $J(s, 0) - 0\dot{J}(s, 0) = 0$  and

$$\left| D_t (J(0, t) - t\dot{J}(0, t)) \right| = t |\ddot{J}(0, t)| \leq C_0 t |J(0, t)| \leq C_0 t |V|$$

because  $|J(0, t)| \leq |J(0, \tau)| = |V|$ . So  $U(t) = J(0, t) - t\dot{J}(0, t)$  vanishes at  $t = 0$ , and we have bounded its derivative. If  $P_{a,b}$  denotes the parallel transport  $T_{\gamma(a)}M \rightarrow T_{\gamma(b)}M$  along  $\gamma = c(0, \cdot)$ , the fundamental theorem of calculus reads for covariant derivatives along a geodesic:

$$U(t) = P_{0,t}U(0) + \int_0^t P_{r,t}\dot{U}(r) dr,$$

as can be easily seen by taking a parallel frame along the geodesic. This gives  $|J(0, t) - t\dot{J}(0, t)| \leq \frac{1}{2}C_0 t^2 |V|$ . Since  $J(0, \tau) = V$ , the first claim is proven.  $\square$

**Lemma 25.** *Suppose  $C_0 \tau^2 < 1$ . Then*

$$|D_s \dot{J}(s, t)| \lesssim (C_0 + \tau C_1) |V|^2.$$

*Proof.* Our approach is to derive some differential equation for  $D_s J = \nabla_J J$ , which has boundary values  $D_s J(s, 0) = 0$  and  $D_s J(s, \tau) = 0$  for all  $s$  because  $J(s, 0) = 0$  is constant in  $s$  and  $J(s, \tau) = \frac{d}{ds} \delta(s)$  is the tangent of a geodesic.

Because  $J$  and  $T$  are coordinate vector fields,  $D_s D_t W = D_t D_s W + R(J, T)W$  for every vector field  $W$ . We see that

$$D_s \dot{J} = D_t D_s J + R(J, T)J$$

and if we let  $U = D_s J$ ,

$$|D_s \dot{J}(s, t)| \leq |\dot{U}| + C_0 |T| |J|^2.$$

We can now use the symmetries of  $R$  to get

$$\begin{aligned} D_s \ddot{J} &= D_s D_t D_t J = D_t D_s D_t J + R(J, T) \dot{J} \\ &= D_t D_t D_s J + D_t (R(J, T) J) + R(J, T) \dot{J} \\ &= D_{tt}^2 (D_s J) + \dot{R}(J, T) J + R(\dot{J}, T) J + 2R(J, T) \dot{J}. \end{aligned}$$

The (negative) left-hand side is, due to the Jacobi equation,

$$-D_s \ddot{J} = D_s (R(J, T) T) = (D_s R)(J, T) T + R(D_s J, T) T + R(J, \dot{J}) T + R(J, T) \dot{J}$$

(note  $D_s T = D_t J = \dot{J}$ ). From now on, we consider  $J$  and  $\dot{J}$  as being part of the given data (which is allowed, as we have already sufficiently described their behavior). So we have a linear second-order ODE for  $U$ :

$$\ddot{U} = AU + B,$$

where both sides scale with  $1/\lambda^2$  when  $t$  is replaced by  $\lambda t$ , and the operator norm of  $A$  is bounded through  $\|A\| \leq C_0 |T|^2(s)$ . At  $s = 0$ , we have  $|T|(0) = 1$  and hence  $\|A\| \tau^2 \leq C_0 \tau^2 \leq 1$  and

$$\begin{aligned} |B| &\lesssim C_1 |J|^2 + C_0 |J| |\dot{J}| \\ &\lesssim C_1 |V|^2 + C_0 |V| \left(\frac{1}{\tau} + C_0 \tau\right) |V| \\ &\lesssim (C_0 + \tau C_1) \frac{1}{\tau} |V|^2. \end{aligned}$$

(we have used Lemma 24 and the assumption that  $C_0 \tau^2 \leq 1$ ). Now consider Fermi coordinates to obtain an ordinary differential equation in Euclidean space: They map  $c(0, \cdot)$  to some coordinate line, say  $x_1$ , along which  $g$  is the identity matrix, and all Christoffel symbols vanish. Therefore, for any smooth vector field  $V = \sum V^i \frac{\partial}{\partial x^i}$ , the covariant derivative in direction  $T = \partial_t c$  is just  $\nabla_T V = \frac{\partial V^i}{\partial x^1} \frac{\partial}{\partial x^i}$ . In this frame, our ODE has the coordinate expression

$$\sum \frac{\partial^2 U^i}{(\partial x^1)^2} \frac{\partial}{\partial x^i} = \sum \left( \sum A_j^i U^j + B^i \right) \frac{\partial}{\partial x^i}.$$

As we only need to know the values of  $U$  on  $x = (t, 0, \dots, 0)$ , this gives a differential equation for the components  $U^i$  of the same form as above. The claim on  $U$  and  $\dot{U}$  is then contained in the following lemma.  $\square$

**Lemma 26.** *Consider some  $C^2$  function  $U : [0; \tau] \rightarrow \mathbb{R}^m$  satisfying the linear second-order differential equation  $\ddot{U} = AU + B$  with smooth time-dependent data  $A(t) \in \mathbb{R}^{m \times m}$  and  $B(t) \in \mathbb{R}^m$  as well as boundary conditions  $U(0) = U(\tau) = 0$ . Then, provided that  $\|A(t)\| \tau^2 \leq 1$  everywhere, we have  $|\dot{U}(t)| \lesssim |B| \tau$ .*



*Proof.* Denote the maxima of  $\|A\|$  and  $|B|$  over  $[0, \tau]$  as  $a$  and  $b$  respectively. As  $U$  is  $C^2$ , there is an upper bound  $K$  for  $|U|$  on  $[0, \tau]$ . Let  $\xi_0$  be a point in  $[0, \tau]$  that maximizes  $|U|^2$ , i.e.,

$$|U(\xi_0)|^2 = K^2.$$

Since  $|U(0)| = |U(\tau)| = 0$ , we must have that  $\xi_0 \in (0, \tau)$  and thus at  $\xi_0$  we have

$$0 = \frac{d}{dt} |U|^2 = 2U \cdot \dot{U}.$$

We now see that

$$\begin{aligned} K^2 + \xi_0^2 \left| \dot{U}(\xi_0) \right|^2 &= \left| U(\xi_0) - \xi_0 \dot{U}(\xi_0) \right|^2 \\ &= \left| \int_0^{\xi_0} t \ddot{U} dt \right|^2 \\ &\leq \left( \int_0^{\xi_0} t (aK + b) dt \right)^2 \\ &= \left( \frac{1}{2} \xi_0^2 (aK + b) \right)^2. \end{aligned}$$

It then follows that

$$K \leq \frac{1}{2} \xi_0^2 (aK + b) \leq \frac{1}{2} \tau^2 (aK + b)$$

so

$$K \leq \frac{\tau^2 b}{2(1 - \frac{1}{2} \tau^2 a)} \leq \tau^2 b$$

if  $\tau^2 a \leq 1$ . It also follows that

$$\xi_0 \left| \dot{U}(\xi_0) \right| \leq \frac{1}{2} \xi_0^2 (aK + b),$$

and so

$$\left| \dot{U}(\xi_0) \right| \leq \frac{1}{2} \tau (ab\tau^2 + b) \leq b\tau.$$

We can now estimate  $\left| \dot{U}(\xi) \right|$  at any point as

$$\begin{aligned} \left| \dot{U}(\xi) \right| &\leq \left| \int_{\xi_0}^{\xi} \ddot{U} dt \right| + \left| \dot{U}(\xi_0) \right| \\ &\leq \int_{\xi_0}^{\xi} (aK + b) dt + b\tau \\ &\leq \tau (ab\tau^2 + b) + b\tau \\ &\leq 3b\tau. \end{aligned}$$

□

**Remark 12.** If  $\gamma$  is not parametrized by arclength, we introduce  $\ell := \tau|T|(0) = \mathbf{d}(p, q)$ , and the estimates become

$$|\tau\dot{J}(0, \tau) - V| \lesssim C_0 \ell^2 |V|, \quad |\tau D_s \dot{J}(0, \tau)| \lesssim (C_0 + \ell C_1) \ell |V|^2.$$

**Remark 13.** The motivation for this calculation is as follows. It is well-known that Jacobi fields grow approximately linear: If  $J(t)$  is a Jacobi field along some arclength-parametrized geodesic  $\gamma(t)$  and  $P_{0,t}$  is the corresponding parallel transport from  $T_{\gamma(0)}M$  to  $T_{\gamma(t)}M$  along  $\gamma$ , then

$$|J(t) - P_{0,t}(J(0) + t\dot{J}(0))| \leq C_0 t^2 (|J(0)| + t|\dot{J}(0)|) \quad \text{for } C_0 t^2 < \pi. \quad (15)$$

In fact, [Jos11, thm. 5.5.3] proves that

$$|J(t) - P_{0,t}(J(0) + t\dot{J}(0))| \leq |J(0)|(\cosh ct - 1) + \frac{d}{dt}|J(0)|\left(\frac{1}{c} \sinh ct - t\right)$$

for  $c = \sqrt{C_0}$ . By Taylor expansion and  $\frac{d}{dt}|J| \leq |\dot{J}|$ , this estimate implies (15). This is the initial-value estimate corresponding to our boundary-value setting.

**Remark 14.** The proofs of Lemmas 24 and 25 can be extended to prove more general lemmas stating that if the derivatives of the curvature tensor up to order  $k$  are bounded by constants  $C_0, \dots, C_k$ , then we can find bounds on  $|D_{t\dots t}^{k+2} J|$  and  $|D_{s\dots s}^k \dot{J}|$ . This is done by noticing that Theorem 23 is true for  $s$  different from zero as well since  $J(s, \cdot)$  is a Jacobi field, and then by differentiating the Jacobi Equation and estimating ODE. (One technical aspect of this approach is that we need to estimate  $|T|(s)$ , which is close to 1. This estimate can be obtained using the fact that  $D_s T = D_t J$  and using Lemma 24.)

## 5.2 Derivatives of the Squared Distance Function

**Definition 15.** For  $p \in M$ , let  $\mathbf{d}_p$  be the geodesic distance to  $p$ ,

$$Y_p := \text{grad } \mathbf{d}_p, \quad X_p := \frac{1}{2} \text{grad } \mathbf{d}_p^2 = \mathbf{d}_p Y_p.$$

**Remark 16.** **\langle a \rangle** As  $Y_p$  is the tangent of an arclength-parametrized geodesic,  $\nabla_{Y_p} Y_p = 0$ . Along this geodesic,  $\mathbf{d}_p$  is exactly the arclength, so  $Y_p(\mathbf{d}_p) = 1$  and  $X_p(\mathbf{d}_p) = \mathbf{d}_p$ .

**\langle b \rangle** The fact  $V(\mathbf{d}_p) = 0$  for  $V \perp Y_p$  is usually referred to as the Gauss lemma.

**\langle c \rangle**  $X_p|_q$  is the tangent  $\dot{\gamma}(1)$  of the geodesic with  $\gamma(0) = p$  and  $\gamma(1) = q$ . Reversing the direction of  $\gamma$  shows  $X_p|_q = -P_{0,1}X_q|_p$ , where  $P$  is the parallel transport along  $\gamma$ . So

$$\exp_p(-X_q|_p) = q, \quad (\exp_p)^{-1}(q) = -X_q|_p = P_{1,0}X_p|_q.$$

**Lemma 27.** For  $V \in T_q M$ , where  $q$  is in a convex neighborhood of  $p$ , recall the Jacobi field  $J$  introduced at the beginning of the section. Then

$$\nabla_V X_p = \tau \dot{J}(0, \tau), \quad \nabla_{V,V}^2 X_p = \tau D_s \dot{J}(0, \tau).$$

*Proof.* For the variation of geodesics  $c$  inducing  $J$ , the  $t$ -derivative is

$$\partial_t c(s, t) = \frac{1}{\tau} P_{0,t}(\exp_p)^{-1} \delta(s) = \frac{1}{\tau} P_{\tau,t} X_p|_{\delta(s)}$$

and hence  $\tau \dot{J}(0, \tau) = \tau D_t \partial_s c(0, \tau) = \tau D_s \partial_t c(0, \tau) = D_s X_p|_{c(0,\tau)} = \nabla_{J(0,\tau)} X_p$ . Differentiating this once more gives the claim for the second derivative. If  $V$  is parallel to  $X_p$ , then use  $\nabla_Y Y = 0$ .  $\square$

**Remark 17.** Analogous to  $(\exp_p)^{-1}(q) = -X_q|_p$ , the derivatives of  $X$  and  $\exp$  correspond to the following:  $\nabla_V X_p$  is the derivative of some Jacobi field with prescribed start and end value, whereas  $d(\exp_p V)(W)$  is the end value  $J(1)$  of a Jacobi field  $J$  along the geodesic  $t \mapsto \exp_p tV$  with  $J(0) = 0$  and  $\dot{J}(0) = W$ , cf. [Kar89, eqn. 1.2.5].

We may now use the estimates in Lemmas 24 and 25 to show the following.

**Corollary 28.** *If  $R$  is the Riemannian curvature tensor of  $(M, g)$ , assume  $\|R\| \leq C_0$  and  $\|\nabla R\| \leq C_1$  everywhere. Let  $q$  be in a convex neighborhood of  $p$  with distance  $\tau$  to  $p$ . Then for  $V \in T_q M$ , we have that*

$$\begin{aligned} |\nabla_V X_p - V| &\lesssim C_0 \tau^2 |V| & \text{if } \tau < \frac{\pi}{2\sqrt{C_0}}, \\ |\nabla_{V,V}^2 X_p| &\lesssim (C_0 + \tau C_1) \tau |V|^2 & \text{if also } C_0 \tau^2 < 1. \end{aligned}$$

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